Analytic vortex solutions in an unusual Mexican-hat potential

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We introduce an unusual Mexican-hat potential, a piecewise parabolic one, and we show that its vortex solutions can be found analytically, in contrast to the case of the standard $|\Psi|^4$ field theory.
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Spontaneous symmetry breaking has been studied traditionally through the $|\Psi|^4$ potential,

$$V(\Psi)=\lambda(|\Psi|^2-v^2)^2. \tag{1}$$

This potential is usually called a Mexican-hat potential by particle physicists, even though the actual Mexican hat has somewhat different wings. The above potential is relevant not only to particle physics, but also to condensed matter physics. In superfluid helium II, for example, it is used for writing down the free energy, and it leads to a lot of interesting physics.

Of particular interest are the vortex solutions that this potential admits. In the context of superfluid helium, these were studied quite a long time ago [1]. Due to their inherent nonlinearity, the corresponding field equations have to be solved numerically. There are analytic approximations for the vortex solutions [2], but the exact solutions can only be found numerically [3].

Vortices play an important role in many fields of physics. It may be of interest then to have analytic vortex solutions, that could be used in further calculations involving vortices. Such analytic solutions would also be of pedagogical interest, since they would show explicitly the various properties of a generic vortex solution.

In this paper we study an unusual Mexican-hat potential that admits vortex solutions. These vortex solutions are analytic though, and are expressed in terms of standard special functions of mathematical physics. Not only can one study therefore the properties of the solutions analytically, but one could also use them in other calculations involving vortices.

Our potential is

$$V(\Psi)=\lambda(|\Psi|-v)^2. \tag{2}$$

This potential is a Mexican-hat potential, but with a kink at $\Psi=0$. It is a piece-wise parabolic potential. Such potentials, but with real order parameters, have also been used in Ginzburg-Landau theories of oil-water-surfactant mixtures [4], because they lead to analytically solvable equations.

The static free energy is then

$$F=\int d^3x\left[\frac{\hbar^2}{2m}|\nabla\Psi|^2+\lambda(|\Psi|-v)^2\right]. \tag{3}$$

If we measure Ψ in units of v, x, y, and z in units of $1/\gamma$, and F in units of $\lambda v^2/\gamma^3$, where $\gamma^2=2m\lambda/\hbar^2$, then we obtain the dimensionless free energy

$$f=\int d^3x[|\nabla\Psi|^2+(|\Psi|-1)^2]. \tag{4}$$

This is minimized when

$$\nabla^2\Psi=\Psi-\frac{\Psi}{|\Psi|}. \tag{5}$$

This is the basic field equation. We note first that it admits one-dimensional solitonic solutions:

$$\Psi(z)=\frac{z-z_0}{|z-z_0|}\,(1-e^{-|z-z_0|}), \tag{6}$$

z_0 being arbitrary. Both Ψ and $\partial\Psi/\partial z$ are continuous at $z=z_0$ in this solution. It is worthwhile to note that the one-dimensional solitonic solutions of the $|\Psi|^4$ theory are also known analytically, unlike the $|\Psi|^4$ vortex solutions.

Let us now concentrate on the solutions of our potential of Eq. (2) for a single isolated vortex. We are thus looking for solutions

$$\Psi=\psi(r)e^{in\phi}, \tag{7}$$

where $x=r\cos\phi$, $y=r\sin\phi$, n being a positive integer, with $\psi(r)>0$. In that case Eq. (5) reduces to

$$\frac{d^2\psi}{dr^2}+\frac{1}{r}\frac{d\psi}{dr}-\left(\frac{n^2}{r^2}+1\right)\psi=-1. \tag{8}$$

Note that this equation is a linear differential equation, albeit an inhomogeneous one, while the corresponding one in the $|\Psi|^4$ theory is nonlinear. The dimensionless free energy for a cylinder of radius R, and length L along the z axis, becomes

$$f=2\pi L\int_0^R dr\,r\left[\left(\frac{d\psi}{dr}\right)^2+\frac{n^2}{r^2}\,\psi^2+(\psi-1)^2\right]. \tag{9}$$

Equation (8) clearly shows that $\psi\to1$ as $r\to\infty$. In particular, $\psi\approx1-(n^2/r^2)$ in that limit. Hence the free energy contains a piece proportional to n^2 that diverges logarithmically as $R\to\infty$. For large R then it is clear that the solution $n=1$ is best, as is usually the case for isolated vortices.

It is interesting to examine the behavior of ψ for small r. Equation (8) easily yields that ψ is linear in r for small r, if $n=1$, while $\psi\approx r^2/(n^2-4)$ if $n>2$. We have then an important difference between our potential and the $|\Psi|^4$ theory

when it comes to highly quantized vortices. Namely, while $\psi \propto r^n$ in the $|\Psi|^4$ theory, here $\psi \propto r^2$ for $n>2$.

Let us now multiply Eq. (8) by $r\psi$ and then integrate from 0 to R. We shall find a relation that enables us to write the free energy in the exact form

$$\frac{f}{2\pi L}=\left[r\psi\,\frac{d\psi}{dr}\right]_0^R+\int_0^R dr\,r(1-\psi). \tag{10}$$

This equation is valid for all n.

We now present the solutions of Eq. (8), beginning with the case where n is a positive odd integer. Let us define first the function

$$g(r)=1-\frac{n}{\sin(n\pi/2)}\int_0^{\pi/2} e^{-r\,\cos\theta}\cos n\theta\; d\theta, \tag{11}$$

with $n=1,3,5,\ldots$. Note that $g\to 1$ as $r\to\infty$, and that $g(0)=0$. In particular, when r is small, $g\approx \pi r/4$ for $n=1$, and $g\approx r^2/(n^2-4)$ for $n=3,5,7,\ldots$. We can easily prove the identity

$$\frac{n^2}{r^2}=\frac{n}{r\,\sin(n\pi/2)}\int_0^{\pi/2}\frac{\partial}{\partial\theta}\left[e^{-r\,\cos\theta}\left(\sin\theta\,\cos n\theta + \frac{n}{r}\sin n\theta\right)\right]d\theta. \tag{12}$$

Its right hand side is equal to

$$\begin{aligned}&\frac{n}{\sin(n\pi/2)}\int_0^{\pi/2} e^{-r\,\cos\theta}\left(\cos n\theta-\cos^2\theta\,\cos n\theta + \frac{\cos\,\theta}{r}\cos\,n\theta+\frac{n^2}{r^2}\cos\,n\theta\right)\\ &\quad=1-g+\frac{d^2g}{dr^2}+\frac{1}{r}\frac{dg}{dr}+\frac{n^2}{r^2}(1-g).\end{aligned} \tag{13}$$

Equations (12) and (13) imply then that g satisfies Eq. (8). It is thus a particular solution of Eq. (8). The most general solution is $g(r)+c_1K_n(r)+c_2I_n(r)$, in terms of modified Bessel functions. However, K_n diverges at the origin and I_n diverges at infinity, while g is well behaved at both limits. Thus we must have $c_1=c_2=0$. Hence

$$\psi(r)=g(r). \tag{14}$$

In particular, for $n=1$ we get

$$\Psi=e^{i\phi}\left(1-\int_0^{\pi/2} e^{-r\,\cos\theta}\cos\theta\; d\theta\right). \tag{15}$$

This is the exact solution representing a singly quantized vortex, and the main result of this paper. It can be written in the form

$$\Psi=\frac{\pi}{2}\,e^{i\phi}[I_1(r)-\mathcal{L}_1(r)], \tag{16}$$

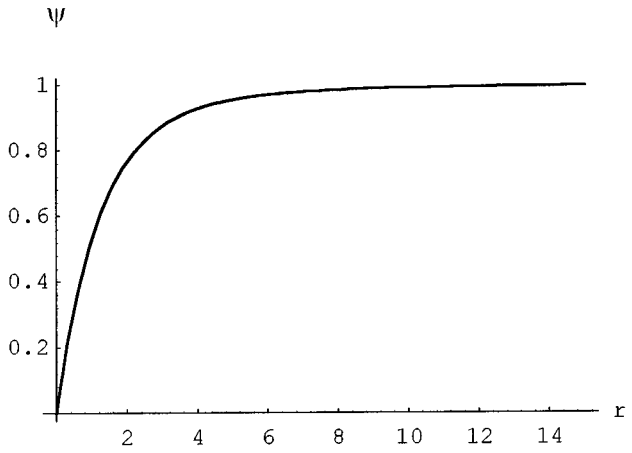

FIG. 1. The solution for a singly quantized isolated vortex.

where $I_1(r)$ is a modified Bessel function and $\mathcal{L}_1(r)$ a modified Struve function [5]. The corresponding $\psi(r)$ is shown in Fig. 1.

Actually, one expects this result, because the function $2\psi(-ir)/\pi$ can be shown to satisfy the standard Struve differential equation [5]. It is thus, in fact, that one gets the idea of using the function of Eq. (11), since the integral representations of the Struve functions involve integrals such as those used in Eq. (11).

We can easily show, using Eq. (15), that $\psi(r)\approx \pi r/4$ for $r\to 0$, for this singly quantized vortex. Furthermore, the asymptotic properties of the modified Struve function give, for $r\to\infty$,

$$\psi(r)\to 1-\frac{1}{r^2}-\frac{3}{r^4}. \tag{17}$$

Thus the behavior of our singly quantized vortex at infinity and at the origin resembles that of the corresponding $|\Psi|^4$ vortex in these limits. Indeed, the ψ for the $|\Psi|^4$ vortex is linear at the origin and tends to $1-(2r^2)^{-1}$ at infinity.

Let us now calculate the free energy, using Eq. (10). The surface term is $2/R^2$, for large R. We can use furthermore the identity [5]

$$r[1-\pi(I_1-\mathcal{L}_1)/2]=\frac{\partial}{\partial r}\left(\frac{\pi r}{2}\,(\mathcal{L}_0-I_0)\right)+\frac{\pi}{2}\,(I_0-\mathcal{L}_0) \tag{18}$$

to get

$$\frac{f}{2\pi L}=\frac{2}{R^2}+\left(\frac{\pi r}{2}\,(\mathcal{L}_0-I_0)\right)_0^R+\frac{\pi}{2}\int_0^R(I_0-\mathcal{L}_0)dr. \tag{19}$$

But we also have [5] the properties $\mathcal{L}_0(0)=0$, $I_0(0)=1$, $\mathcal{L}_0(R)-I_0(R)\approx -2(R^{-1}+R^{-3})/\pi$, and $\int_0^R(I_0-\mathcal{L}_0)dr \approx[2\ln 2R+2\gamma-R^{-2}]/\pi$, where $\gamma=0.577\,215\,7$ is Euler's constant. Using all these, we obtain the final result for the singly quantized isolated vortex:

$$\frac{f}{2\pi L}=\ln R+(\gamma+\ln 2-1)+\frac{1}{2R^2}\approx\ln(1.31R). \tag{20}$$

The corresponding result for the $|\Psi|^4$ singly quantized vortex [1], for a cylinder with dimensionless radius R, is $\ln(1.46R)$.

For vortices with circulation $n=3,5,7,\ldots$, we can find the free energy by combining Eqs. (10) and (14), in which case we get

$$\frac{f}{2\pi L}\approx n\int_0^{\pi/2} d\theta\,\frac{\sin n\theta}{\sin^2\theta}\,[1-e^{-R\,\sin\theta}(1+R\,\sin\theta)]. \tag{21}$$

In order to evaluate this integral, we split it into two pieces, one piece from 0 to ω, and a second piece from ω to $\pi/2$, where ω is a small number such that $\sin\omega\approx\omega$ and $e^{-R\,\sin\omega}\approx 0$. We can then drop the exponential in the $\int_\omega^{\pi/2}$ piece, since R is large, and we can let $\sin\theta\approx\theta$ in the $\int_0^\omega$ piece. Thus

$$\frac{f}{2\pi L}\approx n^2\int_0^\omega \frac{d\theta}{\theta}\,[1-e^{-R\theta}-R\theta e^{-R\theta}]+n\int_\omega^{\pi/2} d\theta\,\frac{\sin n\theta}{\sin^2\theta}. \tag{22}$$

Now the integrals can be done easily, leading to

$$\frac{f}{2\pi L}\approx n^2\left(\ln R+\gamma+\ln 2-\sum_{k=1}^{(n-1)/2}\frac{2}{2k-1}-\frac{1}{n}\right) \tag{23}$$

for $n=3,5,7,\ldots$. For example, $f/2\pi L\approx 9\ln(0.345R)$ for $n=3$, while the corresponding $|\Psi|^4$ vortex [1] has a free energy $f/2\pi L\approx 9\ln(0.38R)$ for a cylinder of length L and dimensionless radius R. For $n=5$, 7, 9 the energies of our vortices are $25\ln(0.20R)$, $49\ln(0.144R)$, and $81\ln(0.11R)$. Of course, the case $n=1$ is the physically interesting one, since it is energetically preferred. We shall find nonetheless the vortex solutions for the case when n is even as well, since the results are simple and analytic.

We examine now the case $n=$even, which is actually simpler than the case of odd n. We can easily show that the function

$$h(r)=n\sum_{k=0}^{n/2}\frac{(n-k-1)!}{k!}\,r^{2k-n}(-1)^{k+n/2}2^{n-2k-1} \tag{24}$$

is a particular solution of Eq. (8), if n is even. The most general solution would be $\psi(r)=h(r)+c_1K_n(r)+c_2I_n(r)$. We must have $c_2=0$, otherwise the solution will diverge at infinity. We also note that the most divergent part of $K_n(r)$ at the origin is $(n-1)!2^{n-1}r^{-n}$, while the most divergent part of $h(r)$ at the origin is $n!r^{-n}(-1)^{n/2}2^{n-1}$. The solution must not have an r^{-n} piece at the origin, hence we must have $c_1=-n(-1)^{n/2}$. Remarkably, this choice ensures that *all* divergences at the origin disappear, leaving us with $\psi(0)=0$, as expected for a vortex. Thus the exact vortex solution for $n=$even is

$$\psi(r)=h(r)-n(-1)^{n/2}K_n(r). \tag{25}$$

For example, the $n=2$ vortex has

$$\psi(r)=1-\frac{4}{r^2}+2K_2(r), \tag{26}$$

while the $n=4$ vortex has

$$\psi(r)=1-\frac{16}{r^2}+\frac{192}{r^4}-4K_4(r). \tag{27}$$

We can now evaluate $f/2\pi L$ using Eq. (10). For the vortex with $n=2$, straight-forward integration using the solution of Eq. (26) yields, for large R,

$$\frac{f}{2\pi L}=[4\ln r+2rK_1(r)+4K_0(r)]_0^R+\frac{8}{R^2}-\frac{32}{R^4}, \tag{28}$$

which reduces to

$$\frac{f}{2\pi L}=4\ln R+4\gamma-4\ln 2-2+\frac{8}{R^2}-\frac{32}{R^4}\approx 4\ln(0.54R). \tag{29}$$

The corresponding result [1] for the $|\Psi|^4$ vortex with $n=2$ is $4\ln(0.59R)$. We note that here too, just as with the cases $n=1$ and 3, the coefficients of R within the logarithm are quite close to those that are appropriate for the $|\Psi|^4$ vortices. In detail, our vortices have the coefficients 1.31, 0.54, 0.345 for $n=1$, 2, 3 respectively, while the $|\Psi|^4$ vortices have 1.46, 0.59, 0.38.

Let us calculate the free energy of the other $n=$even vortices as well, using Eqs. (10) and (25). The integral $\int_0^R rK_n(r)dr$ is needed. We can easily show that

$$\begin{aligned}\frac{d}{dr}\Bigg(&-rK_{n-1}(r)+2n\sum_{i=1}^{n/2-1}(-1)^iK_{n-2i}(r)\\&+n(-1)^{n/2}K_0(r)\Bigg)\\&=rK_n(r),\end{aligned} \tag{30}$$

whence

$$\frac{f}{2\pi L}\approx -n+n^2\left[\ln R+\gamma-\ln 2-\sum_{i=1}^{n/2-1}\left(\frac{n}{2}-i\right)^{-1}\right]. \tag{31}$$

The divergences that the various modified Bessel functions of Eq. (30) exhibit at the origin cancel, confirming thus the correct evaluation of the free energy.

We have thus completed the evaluation of the exact isolated vortex solutions for our unusual Mexican-hat potential. We have been able to obtain analytic solutions for any circulation n, odd or even. This success is due to the fact that the field equation is not nonlinear. It is a linear inhomogeneous equation, and it can therefore be solved exactly. We find the usual features that a vortex must have, i.e., that ψ vanishes at the origin, while it tends to a constant at infinity. The free energy depends logarithmically on the radius of the vortex, as expected. Finally, the actual values of the free energy are quite close to those of the free energy for the standard $|\Psi|^4$ vortex.

Aside from the pedagogical value of these solutions, especially when discussing spontaneous symmetry breaking, our results may be used in more elaborate calculations involving vortices, in many fields of physics.

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